Characteristic functions and Hamilton-Cayley theorem for left eigenvalues of quaternionic matrices*

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Abstract

We introduce the notion of characteristic function of a quaternionic matrix, whose roots are the left eigenvalues. We prove that for all 2×2 matrices and for 3×3 matrices having some zero entry outside the diagonal there is a characteristic function which is a polynomial. For the other 3×3 matrices the characteristic function is a rational function with one point of discontinuity. We prove that Hamilton-Cayley theorem holds in all cases.

Keywords: quaternion, left eigenvalue, characteristic function, Hamilton - Cayley theorem

MSC: 15A33, 15A18

1 Introduction

Very little is known about left eigenvalues of $n \times n$ quaternionic matrices. F. Zhang's papers [9, 10] review their main properties as well as some pathological examples, see also [3]. For $n = 2$ the explicit computation of the left spectrum is due to L. Huang and W. So [4], while the authors studied the symplectic group in [5, 6].

In 1985, R. M. W. Wood [8] proved, by using homotopic methods, that every quaternionic matrix has at least one left eigenvalue. At the end of his paper, Wood notes that "in the 2×2 case of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ there is a

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partially defined determinant $b - ac^{-1}d$ and partially defined characteristic equation

$$\lambda c^{-1}\lambda - \lambda c^{-1}d - ac^{-1}\lambda - b + [ac^{-1}d] = 0 \tag{1}$$

which reduces the eigenvalue problem to the fundamental theorem [of algebra]. The difficulties start with 3×3 matrices".

In this paper we introduce a definition of characteristic function for a quaternionic matrix, which generalizes the usual characteristic polynomial in the real and complex setting. In particular, its roots are the left eigenvalues. Explicitly, we say that $\mu\colon \mathbb{H} \to \mathbb{H}$ is a characteristic function of the matrix $A \in \mathcal{M}(n, \mathbb{H})$ if, up to a constant, its norm verifies that $|\mu(\lambda)| = \mathrm{Sdet}(A - \lambda I)$ for all $\lambda \in \mathbb{H}$, where $\mathrm{Sdet}\colon \mathcal{M}(n, \mathbb{H}) \to [0, +\infty)$ is Study's determinant. As we shall see, this definition fits naturally with Equation (1), as well as with the method proposed by W. So in [7] to compute the left eigenvalues when $n = 3$.

Then we discuss Hamilton-Cayley theorem in this setting. Our main result is as follows.

Theorem A. *For any quaternionic matrix $A \in \mathcal{M}(n, \mathbb{H})$, $n \leq 3$, there exists a characteristic function μ whose extension to a map $\mu\colon \mathcal{M}(n, \mathbb{H}) \to \mathcal{M}(n, \mathbb{H})$ verifies Hamilton-Cayley, that is $\mu(A) = 0$.*

For $n = 2$, a characteristic function like that in (1) is a polynomial $\mu(\lambda)$ for which it is easy to check that $\mu(A) = 0$. It follows that

$$Ac^{-1}A = Ac^{-1}d + ac^{-1}A + (b - ac^{-1}d)I,$$

which generalizes the well known formula $A^2 = (\mathrm{tr}A)A - (\det A)I$ in the commutative setting. When $n = 3$ and the matrix has some zero entry outside the diagonal, we shall find a polynomial characteristic function that verifies Hamilton-Cayley. Otherwise, there is a characteristic function which is, outside a point of discontinuity, a rational function. We are able to extend it to a map $\mu\colon \mathcal{M}(n, \mathbb{H}) \to \mathcal{M}(n, \mathbb{H})$ and we prove by brute force that Hamilton-Cayley is verified too.

At the end of the paper we discuss another possible definition of characteristic function.

2 Preliminaries

We consider the quaternionic space $\mathbb{H}^n$ as a *right* vector space over $\mathbb{H}$. Two square matrices $A, B \in \mathcal{M}(n, \mathbb{H})$ are *similar* if $B = PAP^{-1}$ for some invertible square matrix P.

If A is a quaternionic $n \times n$ matrix, let us write $A = X + jY$, with $X, Y \in \mathcal{M}(n, \mathbb{C})$, and let

$$c(A) = \begin{bmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{bmatrix} \in \mathcal{M}(2n, \mathbb{C})$$

be its *complex form*. We have $c(A \cdot B) = c(A) \cdot c(B)$, $c(A + B) = c(A) + c(B)$ and $c(tA) = tc(A)$ if $t \in \mathbb{R}$. In particular, A is invertible if and only if $c(A)$ is invertible. Moreover, $\det c(A) \geq 0$ is a nonnegative real number, so we can define the *Study's determinant* of A as

$$\operatorname{Sdet}(A) = (\det c(A))^{1/2} \geq 0. \tag{2}$$

For complex matrices, Sdet equals the norm of the complex determinant, see [1, 2] for a general discussion of quaternionic determinants. The following properties are immediate:

1. $\operatorname{Sdet}(A \cdot B) = \operatorname{Sdet}(A) \cdot \operatorname{Sdet}(B)$;
2. A is invertible if and only if $\operatorname{Sdet}(A) > 0$;
3. if A, B are similar matrices then $\operatorname{Sdet}(A) = \operatorname{Sdet}(B)$.

We also need the following result.

Lemma 2.1. *For a matrix with boxes M, N of size $m \times m$ and $n \times n$ respectively we have*

$$\operatorname{Sdet} \begin{bmatrix} 0 & M \\ N & * \end{bmatrix} = \operatorname{Sdet}(M) \cdot \operatorname{Sdet}(N).$$

It follows that $\operatorname{Sdet}(A) = |q_1 \cdots q_n|$ when A is a triangular matrix, with $q_1, \ldots, q_n$ being the elements of the diagonal.

Sometimes we shall permute two columns or rows of the matrix A. Or we shall add to a column a right linear combination of the columns. This will not affect the value of the determinant because the matrices of the type $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $P = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}$ verify $\operatorname{Sdet}(P) = 1$.

Remark 2.2. Up to the exponent $1/2$ in (2), this is the same determinant that the one in Theorem 8.1 of [9] that we shall refer to later in Sect. 7. The exponent is normalized in order to have $\operatorname{Sdet}(A) = |q_1 \cdots q_n|$ for a diagonal matrix $A = \operatorname{diag}(q_1, \ldots, q_n)$.

3 Left eigenvalues and characteristic functions

A quaternion $\lambda \in \mathbb{H}$ is said to be a *left eigenvalue* of the matrix $A \in \mathcal{M}(n, \mathbb{H})$ if $Av = \lambda v$ for some vector $v \in \mathbb{H}^n$, $v \neq 0$. Equivalently, the matrix $A - \lambda I$ is not invertible, that is $\mathrm{Sdet}(A - \lambda I) = 0$, where Sdet is Study's determinant defined in Section 2.

Definition 3.1. A map $\mu\colon \mathbb{H} \to \mathbb{H}$ is a *characteristic function* of the matrix $A \in \mathcal{M}(n, \mathbb{H})$ if, up to a constant, $|\mu(\lambda)| = \mathrm{Sdet}(A - \lambda I)$ for all $\lambda \in \mathbb{H}$.

Notice that λ is a left eigenvalue of A if and only if $\mu(\lambda) = 0$.

Remark 3.2. It is well known that the left spectrum is not invariant under similarity. However, if P is a *real* invertible matrix then $\mathrm{Sdet}(PAP^{-1} - \lambda I) = \mathrm{Sdet}(A - \lambda I)$, so A and PAP^{-1} have the same characteristic functions.

Example 3.3. Diagonal and triangular matrices.

If $A = \mathrm{diag}(q_1, \ldots, q_n)$ then $\mu(A) = (q_n - \lambda) \cdots (q_1 - \lambda)$ is a characteristic function. Analogously for triangular matrices.

Example 3.4. 2×2 matrices.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}(2, \mathbb{H})$. If $b = 0$ then $\mathrm{Sdet}(A) = |da|$ and the map $\mu(\lambda) = (d - \lambda)(a - \lambda)$ is a characteristic function. If $b \neq 0$ we have

$$A \sim \begin{bmatrix} 0 & b \\ c - db^{-1}a & d \end{bmatrix}$$

so

$$\mathrm{Sdet}(A) = |b||c - db^{-1}a|.$$

Consequently, we consider the characteristic function

$$\mu(\lambda) = c - (d - \lambda)b^{-1}(a - \lambda). \tag{3}$$

Obviously, the characteristic function of a matrix is not unique. For instance, by permuting rows and columns we can obtain $\mu(\lambda) = b - (a - \lambda)c^{-1}(d - \lambda)$ which is Wood's function in Equation (1) (there is a misprint in the original article). However, as we shall see in Section 4, it is preferable to take minors starting from the top right corner as we do.

4 Characteristic function of 3×3 matrices

Now let $A = \begin{bmatrix} a & b & c \\ f & g & h \\ p & q & r \end{bmatrix}$ be a 3×3 quaternionic matrix. The computation of $\mathrm{Sdet}(A)$ can be done as follows (a similar algorithm is valid for any $n > 3$).

4.1 Case $n = 3, c \neq 0$

First we consider the generic case when $c \neq 0$. In this case we can create zeroes in the first row,

$$A \sim \begin{bmatrix} 0 & 0 & c \\ f - hc^{-1}a & g - hc^{-1}b & h \\ p - rc^{-1}a & q - rc^{-1}b & r \end{bmatrix}.$$

By Lemma 2.1 and the 2×2 case, it follows:

Proposition 4.1. *If $c \neq 0$, then* $\mathrm{Sdet}(A)$ *is given:*

1. *when $g - hc^{-1}b \neq 0$, by*
$$|c| \cdot |g - hc^{-1}b| \cdot |p - rc^{-1}a - (q - rc^{-1}b)(g - hc^{-1}b)^{-1}(f - hc^{-1}a)|;$$
2. *when $g - hc^{-1}b = 0$, by*
$$|c| \cdot |q - rc^{-1}b| \cdot |f - hc^{-1}a|.$$

Corollary 4.2. *Let us call $\lambda_0 = g - hc^{-1}b$ the* pole *of A. Then*

$$\mathrm{Sdet}(A - \lambda_0 I) = |c| \cdot |q - (r - \lambda_0)c^{-1}b| \cdot |f - hc^{-1}(a - \lambda_0)|. \tag{4}$$

By applying Prop. 4.1 and Cor. 4.2 to $A - \lambda I$ we find the following characteristic function of A.

Definition 4.3. When $c \neq 0$, a characteristic function for the 3×3 matrix A can be defined as follows:

1. if $\lambda_0 = g - hc^{-1}b$ is the pole of A,
$$\mu(\lambda_0) = \left(q - (r - \lambda_0)c^{-1}b\right)\left(f - hc^{-1}(a - \lambda_0)\right);$$

2. otherwise,

$$\begin{aligned}\mu(\lambda) \quad = \quad & (\lambda_0 - \lambda)\left[\left(p - (r-\lambda)c^{-1}(a-\lambda)\right) - \right.\\ & \left.\left(q - (r-\lambda)c^{-1}b\right)(\lambda_0 - \lambda)^{-1}\left(f - hc^{-1}(a-\lambda)\right)\right].\end{aligned}$$

Remark 4.4. In [7], W. So proved that the left eigenvalues of a 3×3 matrix can be computed as roots of certain polynomials of degree ≤ 3. Even though our computation is different from his, we obtain that the function in Def. 4.3 is exactly So's formula in [7, p. 563]. This is why we have chosen to compute determinants starting from the top right corner.

4.2 Case $n = 3, c = 0$

We briefly review what happens when $c = 0$. First, if both $b, h = 0$ we have a triangular matrix, then we can take

$$\mu(\lambda) = (r - \lambda)(g - \lambda)(a - \lambda). \tag{5}$$

If $b = 0$ but $h \neq 0$ we can reduce to the 2×2 case by Lemma 2.1, so we take

$$\mu(\lambda) = \left(q - (r-\lambda)h^{-1}(g-\lambda)\right)(a - \lambda). \tag{6}$$

Finally, if $b \neq 0$ we can (see the proof of Theorem 6.3) create a zero in the left top corner of $A - \lambda I$ and then permute the second and last column, in order to reduce the matrix $(A - \lambda I)P$ to the 2×2 case. Alternatively, we can simply permute the second and last column and the second and last row of A, in order to obtain a matrix PAP^{-1} with the same characteristic function, to which Subsection 4.1 applies. Notice however that with the latter method we obtain a rational function, not a polynomial.

5 Continuity

The following example shows that the characteristic function μ in Definition 4.3 may not be continuous, even if its norm $|\mu|$ is a continuous map.

Let

$$A = \begin{bmatrix} 0 & i & 1 \\ 3i - k & 0 & 1 \\ k & -1 + j + k & 0 \end{bmatrix}.$$

Its pole (see Cor. 4.2) is $\lambda_0 = -i$ and

$$\mu(\lambda_0) = (j + k)(2i - k) = 1 - i + 2j - 2k.$$

However, for $\lambda \neq \lambda_0$ we have

$$\mu(\lambda) = (-i-\lambda)\left(k - \lambda^2 - (-1+j+k+\lambda i)(-i-\lambda)^{-1}(3i-k+\lambda)\right),$$

and by taking $\lambda = -i + \varepsilon j$, $\varepsilon \in \mathbb{R}$, with $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon\to 0} \mu(-i+\varepsilon j) = 1 + i + 2j + 2k \neq \mu(\lambda_0).$$

In fact, the limit

$$\lim_{\varepsilon\to 0} \mu(-i+\varepsilon q) = -q(j+k)q^{-1}(2i-k)$$

depends on q, so $\lim_{\lambda\to\lambda_0} \mu(\lambda)$ does not exist.

It is an open question whether it is always possible to find a continuous characteristic function.

6 Hamilton-Cayley theorem

We now discuss Hamilton-Cayley theorem.

6.1 Case $n = 2$

Theorem 6.1. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a* 2×2 *quaternionic matrix. Let* $\mu(\lambda) = c - (d-\lambda)b^{-1}(a-\lambda)$ *be the characteristic function defined in (3). Then* $\mu(A) = 0$.

Proof. We have

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} - \begin{bmatrix} d-a & -b \\ -c & 0 \end{bmatrix} \begin{bmatrix} b^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} 0 & -b \\ -c & a-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

□

Corollary 6.2. $Ab^{-1}A = Ab^{-1}a + db^{-1}A + (c - db^{-1}a)I$.

6.2 Case $n = 3$, $c = 0$

For $n = 3$, a direct computation will show that Hamilton-Cayley theorem is true when $c = 0$ (see Section 4).

Proposition 6.3. *Let* $A = \begin{bmatrix} a & b & 0 \\ f & g & h \\ p & q & r \end{bmatrix}$. *Let* $\mu(\lambda)$ *be the characteristic function defined in Subsection 4.2. Then* $\mu(A) = 0$.

Proof. If $b, h = 0$ we take formula (5), so $\mu(A)$ equals

$$\begin{bmatrix} r-a & 0 & 0 \\ -f & r-g & 0 \\ -p & -q & 0 \end{bmatrix} \begin{bmatrix} g-a & 0 & 0 \\ -f & 0 & 0 \\ -p & -q & g-r \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -f & a-g & 0 \\ -p & -q & a-r \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $b = 0$, $h \neq 0$ we take formula (6), then we check

$$\begin{bmatrix} r-a & 0 & 0 \\ -f & r-g & -h \\ -p & -q & 0 \end{bmatrix} h^{-1} \begin{bmatrix} g-a & 0 & 0 \\ -f & 0 & -h \\ -p & -q & (g-r) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -f & a-g & -h \\ -p & -q & a-r \end{bmatrix} = q \begin{bmatrix} 0 & 0 & 0 \\ -f & a-g & -h \\ -p & -q & a-r \end{bmatrix},$$

that is, $(rI - A)h^{-1}(gI - A)(aI - A) = q(aI - A)$, hence $\mu(A) = 0$.

If $b \neq 0$, we have

$$\mathrm{Sdet}(A - \lambda I) = \mathrm{Sdet} \begin{bmatrix} 0 & 0 & b \\ f - (g-\lambda)b^{-1}(a-\lambda) & h & g-\lambda \\ p - qb^{-1}(a-\lambda) & r-\lambda & q \end{bmatrix},$$

so we are in the 2×2 situation (see Lemma 2.1). First, assume $h = 0$ and let us take $\mu(\lambda) = (r-\lambda)\left(f - (g-\lambda)b^{-1}(a-\lambda)\right)$. We check

$$\begin{bmatrix} r-a & -b & 0 \\ -f & r-g & 0 \\ -p & -q & 0 \end{bmatrix} \begin{bmatrix} g-a & -b & 0 \\ -f & 0 & 0 \\ -p & -q & g-r \end{bmatrix} b^{-1} \begin{bmatrix} 0 & -b & 0 \\ -f & a-g & 0 \\ -p & -q & a-r \end{bmatrix} = \begin{bmatrix} r-a & -b & 0 \\ -f & r-g & 0 \\ -p & -q & 0 \end{bmatrix} f,$$

that is $(rI - A)(gI - A)b^{-1}(aI - A) = (rI - A)f$, hence $\mu(A) = 0$.

On the other hand, if $h \neq 0$ we take

$$\mu(\lambda) = p - qb^{-1}(a-\lambda) - (r-\lambda)h^{-1}\left(f - (g-\lambda)b^{-1}(a-\lambda)\right). \tag{7}$$

Then we compute

$$pI - qb^{-1}(aI - A) - (rI - A)h^{-1}f = \begin{bmatrix} p - (r-a)h^{-1}f & q + bh^{-1}f & 0 \\ qb^{1}f + fh^{-1}f & p - qb^{-1}(a-g) - (r-g)h^{-1}f & qb^{-1}h + f \\ qb^{-1}p + ph^{-1}f & qb^{-1}q + qh^{-1}f & p - qb^{-1}(a-r) \end{bmatrix}$$

and we check it equals

$$-\begin{bmatrix} r-a & -b & 0 \\ -f & r-g & -h \\ -p & -q & 0 \end{bmatrix} h^{-1} \begin{bmatrix} g-a & -b & 0 \\ -f & 0 & -h \\ -p & -q & g-r \end{bmatrix} b^{-1} \begin{bmatrix} 0 & -b & 0 \\ -f & a-g & -h \\ -p & -q & a-r \end{bmatrix} = \\ -(rI-A)h^{-1}(gI-A)b^{-1}(aI-A),$$

hence $\mu(A) = 0$. □

Lemma 6.4. *Let A be a quaternionic matrix such that $\mu(\lambda) = 0$ for some quaternionic polynomial $\mu(\lambda)$. Let $B = PAP^{-1}$ be a similar matrix, with P a real matrix. Then $\mu(B) = 0$.*

Proof. Let $\mu(\lambda) = q_1\lambda q_2\lambda \cdots q_k\lambda q_{k+1}$ be a monomial. Then $\mu(B) = P\mu(A)P^{-1}$. □

Notice that the same result is true when $\mu(\lambda)$ is a rational function.

By permuting rows and columns (see Remark 3.2) we deduce:

Corollary 6.5. *Let A be a 3×3 quaternionic matrix with some zero entry outside the diagonal. Then there exists a polynomial characteristic function μ such that $\mu(A) = 0$.*

Example 6.6. Let us consider the matrix $A = \begin{bmatrix} 1 & i & i \\ i & j & k \\ 0 & -1 & j \end{bmatrix}$. It is real similar to $\begin{bmatrix} j & -1 & 0 \\ k & j & i \\ i & i & 1 \end{bmatrix}$, whose characteristic function is given by formula (7), that is

$$\mu(\lambda) = i + i(j-\lambda) + (1-\lambda)i\left(k + (j-\lambda)^2\right).$$

Then the following equation holds:

$$-AiA^2 + AiAj + AkA + iA^2 - iAj + A(i+j) - (i+k)A + (k-j)I = 0.$$

6.3 Case $n = 3, c \neq 0$

When $c \neq 0$, the characteristic function of the matrix A is a rational function with a pole. We shall extend it to a map in the space of matrices in the following natural way.

Let $\lambda_0 = g - hc^{-1}b$ be the pole of A. Let

$$f_0 = f - hc^{-1}(a - \lambda_0),$$

$$q_0 = q - (r - \lambda_0)c^{-1}b.$$

Lemma 6.7. *The matrix $\lambda_0 I - A$ is invertible if and only if $f_0, q_0 \neq 0$.*

Proof. By Corollary 4.2, $\mathrm{Sdet}(\lambda_0 I - A) = |c||q_0 f_0|$. □

Definition 6.8. We define $\mu\colon \mathcal{M}(n,\mathbb{H}) \to \mathcal{M}(n,\mathbb{H})$ as follows (see Definition 4.3):

1. if $\lambda_0 I - B$ is invertible, then $\mu(B) = q_0 f_0 I$;
2. otherwise,

$$\begin{aligned}\mu(B) \quad = \quad & (\lambda_0 I - B)\left[\left(pI - (r - B)c^{-1}(aI - B)\right) - \right. \\ & \left.\left(qI - (rI - B)c^{-1}b\right)(\lambda_0 - B)^{-1}\left(fI - hc^{-1}(aI - B)\right)\right].\end{aligned}$$

The following Proposition completes the proof of Theorem A.

Proposition 6.9. *The map μ in Def. 6.8 satisfies Hamilton-Cayley theorem, that is $\mu(A) = 0$*

Proof. If $\lambda_0 I - A$ is not invertible, then $\mu(A) = q_0 f_0 I = 0$ by Lemma 6.7. Otherwise it suffices to prove that

$$pI - (rI - A)c^{-1}(aI - A) \tag{8}$$

equals

$$\left(qI - (rI - A)c^{-1}b\right)(\lambda_0 I - A)^{-1}\left(fI - hc^{-1}(aI - A)\right). \tag{9}$$

Lemma 6.10. *A direct computation shows that the first term (8) is*

$$\begin{bmatrix} -bc^{-1}f & -q + (r-a)c^{-1}b + bc^{-1}(a-g) & -bc^{-1}h \\ +(r-g)c^{-1}f + hc^{-1}p & p - fc^{-1}b - hc^{-1}q - (r-g)c^{-1}(a-g) & -f + (r-g)c^{-1}h + hc^{-1}(a-r) \\ -qc^{-1}f & -pc^{-1}b + qc^{-1}(a-g) & -qc^{-1}h \end{bmatrix}.$$

We now want to compute the term (9).

We start by computing $(\lambda_0 I - A)^{-1}$ by Gaussian elimination.

Let

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c^{-1}(\lambda_0 - a) & 0 & 1 \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c^{-1}b & 1 \end{bmatrix}$$

Then

$$(\lambda_0 I - A)P_1P_2 = \begin{bmatrix} 0 & 0 & -c \\ -f_0 & 0 & -h \\ -p^* & -q_0 & \lambda_0 - r \end{bmatrix}, \tag{10}$$

where

$$p^* = p - (\lambda_0 - r)c^{-1}(\lambda_0 - a).$$

The inverse of the matrix $(\lambda_0 I - A)P_1P_2$ in (10) can be computed by hand; it is

$$B = \begin{bmatrix} f_0^{-1}hc^{-1} & -f_0^{-1} & 0 \\ -q_0^{-1}n^* & q_0^{-1}p^*f_0^{-1} & -q_0^{-1} \\ -c^{-1} & 0 & 0 \end{bmatrix}$$

where

$$n^* = p^*f_0^{-1}hc^{-1} - (\lambda_0 - r)c^{-1}.$$

It follows that

$$(\lambda_0 I - A)^{-1} = P_1P_2B =$$

$$\begin{bmatrix} f_0^{-1}hc^{-1} & -f_0^{-1} & 0 \\ -q_0^{-1}n^* & q_0^{-1}p^*f_0^{-1} & -q_0^{-1} \\ c^{-1}(\lambda_0 - a)f_0^{-1}hc^{-1} + c^{-1}bq_0^{-1}n^* - c^{-1} & -c^{-1}(\lambda_0 - a)f_0^{-1} - c^{-1}bq_0^{-1}p^*f_0^{-1} & +c^{-1}bq_0^{-1} \end{bmatrix}.$$

Moreover

$$F = fI - hc^{-1}(aI - A) \quad = \quad \begin{bmatrix} f & hc^{-1}b & h \\ hc^{-1}f & f - hc^{-1}(a-g) & hc^{-1}h \\ hc^{-1}p & hc^{-1}q & f - hc^{-1}(a-r) \end{bmatrix},$$

while

$$Q = qI - (rI - A)c^{-1}b = \quad = \quad \begin{bmatrix} q - (r-a)c^{-1}b & bc^{-1}b & b \\ fc^{-1}b & q - (r-g)c^{-1}b & hc^{-1}b \\ pc^{-1}b & qc^{-1}b & q \end{bmatrix}.$$

We have to compute (9), that is QP_1P_2BF.

First we compute $(P_1P_2B)F$. For instance, its first column is given by

$$[(P_1P_2B)F]^1 = \begin{bmatrix} 0 \\ -q_0^{-1}n^*f + q_0^{-1}p^*f_0^{-1}hc^{-1}f - q_0^{-1}hc^{-1}p \\ +c^{-1}bq_0^{-1}n^*f - c^{-1}f - c^{-1}bq_0^{-1}p^*f_0^{-1}hc^{-1}f + c^{-1}bq_0^{-1}hc^{-1}p \end{bmatrix}.$$

Now we check for instance the entry $(1,1)$ of the matrix $Q(P_1P_2BF)$. We have

$$\begin{array}{rl} [Q(P_1P_2BF)]_1^1 & = \\ bc^{-1}b\left(-q_0^{-1}n^*f + q_0^{-1}p^*f_0^{-1}hc^{-1}f - q_0^{-1}hc^{-1}p\right) & + \\ b\left(+c^{-1}bq_0^{-1}n^*f - c^{-1}f - c^{-1}bq_0^{-1}p^*f_0^{-1}hc^{-1}f + c^{-1}bq_0^{-1}hc^{-1}p\right) & = \\ -bc^{-1}f & \end{array}$$

which indeed is the entry $(1,1)$ in Corollary 6.10.

The other entries are computed in a similar way. $\square$

Example 6.11. Let $A = \begin{pmatrix} 1 & i & -j \\ i & -1 & k \\ 1 & -1 & j \end{pmatrix}$. The pole is $\lambda_0 = -2$ and $\mu(\lambda_0) = -5 + 8j$. For $\lambda \neq 2$, the characteristic function is

$$\mu(\lambda) = -(2+\lambda)\left(2 + \lambda(-1+j) - \lambda j \lambda + (-1+i-\lambda k)(2+\lambda)^{-1}i(2-\lambda)\right).$$

With the notations of the of proof of Proposition 6.9, it is

$$(\lambda_0 I - A)^{-1} = (1/12)\begin{pmatrix} -3 & 3i & 0 \\ 2i-j-k & -8+2i+j+3k & 2+2i+4k \\ 1+i-j & -3-i+2j+k & 2-2+j+k \end{pmatrix}.$$

$$P = \begin{pmatrix} -j & 1-i+3k & 1 \\ -k & 3-i-3j & -i \\ -k & -2j+k & i \end{pmatrix},$$

$$Q = \begin{pmatrix} -1+i-k & j & i \\ j & -1+i+k & 1 \\ -k & k & -1 \end{pmatrix}$$

and

$$F = \begin{pmatrix} i & 1 & k \\ 1 & 3i & j \\ -i & i & 2i-k \end{pmatrix}.$$

We have $P - Q(\lambda_0 I - A)^{-1}F = 0$.

7 Final remarks

In this Section we discuss a different approach to the definition of characteristic functions for left eigenvalues.

In order to clarify concepts, let us briefly comment the same problem but for *right* eigenvalues. Let $c(A) \in \mathcal{M}(2n, \mathbb{C})$ be the complex form of the matrix $A \in \mathcal{M}(n, \mathbb{H})$ (see Section 2). Then, as it is well known, the right eigenvalues of A are the quaternions qzq^{-1}, where $q \in \mathbb{H}$, $q \neq 0$, and z is a complex eigenvalue of $c(A)$. It follows:

Theorem 7.1 ([9])**.** *Let* $p(z) = \det(c(A) - zI) = \sum_{k=0}^{2n} c_k z^k$*,* $c_k \in \mathbb{R}$*, be the characteristic polynomial of* $c(A)$*. Then* $p(A) = \sum_{k=0}^{2n} c_k A^k = 0$.

Now, let $\lambda = x + jy$, with $x, y \in \mathbb{C}$, be a left eigenvalue of A. Equivalently, the matrix $c(A - \lambda I)$ is not invertible. It follows that the left eigenvalues are the roots of the function $\sigma\colon \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ given by

$$\sigma(x, y) = \det \begin{bmatrix} X - xI & -\overline{Y} + \overline{y}I \\ Y - yI & \overline{X} - \overline{x}I \end{bmatrix}. \tag{11}$$

Let $A = X + jY$, with $X, Y \in \mathcal{M}(n, \mathbb{C})$. Then Hamilton-Cayley theorem could be stated as $\sigma(X, Y) = 0$, provided this has a meaning. However we have the following counterexample even for $n = 2$.

Example 7.2. Let $A = \begin{bmatrix} 0 & i \\ j & 0 \end{bmatrix}$. Let $x = x_1 + ix_2$, $y = y_1 + iy_2$. Then

$$\sigma(x, y) = 1 + (x_1^2 + x_2^2 + y_1^2 + y_2^2)^2 - 4x_2y_1.$$

On the other hand, it is $X = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, so $X_1 = 0$, $X_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Y_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $Y_2 = 0$, then $\sigma(X, Y) = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$.

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